Assignment 4.

This assignment is due April 4th. If you need more time, ask for an extension (just don't get overwhelmed by homeworks piling up).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Each of the following expression defines a function D on the set of 3×3 matrices over the field of real numbers. In which of the these is D a 3-linear function (that is, a 3-linear function as a function of rows of matrix)?
  (a) $D(A) = A_{11} + A_{21} + A_{33}$,
  (b) $D(A) = (A_{11})^2 + 3A_{11}A_2$,
  (c) $D(A) = A_{11}A_{12}A_{33}$,
  (d) $D(A) = A_{13}A_{22}A_{32} + 5A_{12}A_{22}A_{32}$,
  (e) $D(A) = 0$,
  (f) $D(A) = 1$.

(2) Use Cramer's rule to solve system of equations:

$$\begin{array}{ccccc} 3x & - & 2y & = & 7 \\ 3y & - & 2z & = & 6 \\ 3z & - & 2x & = & -1 \end{array}$$

(3) Let c be an arbitrary element of a field F. Give an example of matrices A, B over F such that $\det A = \det B = 0$, but $\det(A + B) = c$.

Comment. This shows that, while $\det AB$ is easily expressed through $\det A$ and $\det B$, there *can be no* expression of $\det(A + B)$ through $\det A$ and $\det B$.

(4) (a) If F is a field and $A \in F^{3\times 3}$ is given by

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

show that $\det A = 0$.

  (b) An $n \times n$ matrix A over a field F is called *skew-symmetric* if $A^t = -A$. If A is a skew-symmetric matrix with complex entries (or, more generally, $\operatorname{char} F \neq 2$), and n is *odd*, prove that $\det A = 0$. (Hint: use $\det A = \det A^t$.)

  (c) An $n \times n$ matrix A over a field F is called *orthogonal* if $A^t A = 1$. If A is orthogonal, show that $\det A = \pm 1$.

(5) Let A_n be an $n \times n$ matrix of the form

$$\begin{pmatrix} 1 & 1 & 0 & & \dots & & 0 \\ 1 & 1 & 1 & 0 & \dots & & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ & 0 & & \ddots & & & \vdots \\ \vdots & \vdots & & & \ddots & & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 & 1 \\ 0 & 0 & & \dots & 0 & 1 & 1 \end{pmatrix}$$

(that is, entries on the main diagonal and those adjacent to the main diagonal are equal to 1, all other entries are equal to 0). Find recursive formula for $\det A_n$. Hint: using row (or column) decomposition of determinant, reduce computing $\det A_n$ to computing $\det A_{n-1}$ and $\det A_{n-2}$.

Comment. Now, if one wants an explicit formula for $\det A_n$, one can use problem 6 from HW2.

— *see next page* —

(6) Let V be the vector space of $n \times n$ matrices over the field F. Let $B \neq 0$ be a fixed element in V and let T_B be a linear operator $V \to V$ such that $T_B(A) = AB - BA$. Show that $\det T_B = 0$. (Hint: show that T_B has non-trivial nullspace.)

(7) (Vandermonde determinant and Lagrange interpolation.)

(a) Prove that the determinant of the Vandermonde matrix

$$\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

is $(b-a)(c-a)(c-b)$.

(b) Prove that the determinant of the $(n+1) \times (n+1)$ Vandermonde matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 1 & x_{n+1} & x_{n+1}^2 & \dots & x_{n+1}^n \end{pmatrix}$$

is

$$\prod_{n+1 \geq i > j \geq 1} (x_i - x_j).$$

(Hint: there are multiple ways of doing this. Straightforward way is to use induction: subtract first row from all other rows. Bring out brackets $(x_i - x_1)$. Decompose determinant in the first column. Show that in the resulting $n \times n$ determinant, one can get rid of all instances of x_1, using column transformations.
Less straightforward way is to argue that the determinant is a polynomial $p(x_1)$ in x_1 with coefficients that depend on x_i, $i \geq 2$. Argue that $\deg p \leq n$, find n roots of p. Find leading coefficient of p. Argue that that's enough to uniquely define the polynomial p.)

(c) Show that if $x_1, x_2, \dots, x_{n+1}$ are distinct numbers, and $y_1, \dots, y_{n+1}$ are arbitrary numbers, then the linear system

$$\begin{array}{ccccccccccc} a_0 & + & a_1 x_1 & + & a_2 x_1^2 & + & \dots & + & a_n x_1^n & = & y_1 \\ a_0 & + & a_1 x_2 & + & a_2 x_2^2 & + & \dots & + & a_n x_2^n & = & y_2 \\ \vdots & & & & & & & & & & \vdots \\ \vdots & & & & & & & & & & \vdots \\ a_0 & + & a_1 x_{n+1} & + & a_2 x_{n+1}^2 & + & \dots & + & a_n x_{n+1}^n & = & y_{n+1} \end{array}$$

with unknowns a_i has a unique solution. (Hint: what is the matrix of this system? When is its determinant equal to 0?)

(d) Let $x_1, x_2, \dots, x_n$ be distinct numbers, and $y_1, \dots, y_{n+1}$ be arbitrary numbers. Show that there is a unique polynomial of degree $\leq n$ $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ such that

$$f(x_i) = y_i, \quad i = 1, \dots, n+1.$$

Hint: see the previous item.
Comment. This is an alternative proof of Lagrange Interpolation theorem. Advantage over the proof I gave in one of the first lectures is that, modulo Vandermonde determinant, the solution does not require any computations. Disadvantage is that a) in these terms, it's not as easy to find f explicitly; b) this way requires the notion of determinant.